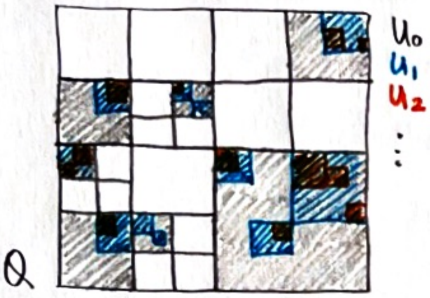
§. Reverse Hölder Property of A_p Weights

Let $w \in A_p$, $1<p<\infty$. Then there exist constants $C, \gamma > 0$ (depending on $n, p, [w]_{A_p}$) such that for every cube Q:

$$\boxed{\left(\frac{1}{|Q|}\int_Q w^{1+\gamma}\right)^{\frac{1}{1+\gamma}} \le C\langle w\rangle_Q}$$

Proof: Fix a cube Q, and let $\mathcal{D}(Q)$ denote the collection of cubes starting with Q, then all the subcubes of Q obtained by dividing Q into a mesh of 2^{ni} subcubes of length $\frac{1}{2^i}\ell(Q)$, for $i \in \mathbb{N}$.



Define an increasing sequence: $\alpha_0 < \alpha_1 < \dots$ by:

$\alpha_0 := \langle w\rangle_Q$

$\alpha_1 := \frac{2^n}{\alpha}\langle w\rangle_Q$ where $\alpha \in (0,1)$ is to be chosen later.

$\vdots$

$\alpha_k = \left(\frac{2^n}{\alpha}\right)^k \langle w\rangle_Q$

$\vdots$

For all $k \ge 0$, define the collections & sets:

$\mathcal{S}_0 := \{\text{maximal } R \in \mathcal{D}(Q) : \langle w\rangle_R > \alpha_0 = \langle w\rangle_Q\} = \{Q_j^0\}_j$; $U_0 := \bigcup_j Q_j^0$

$\mathcal{S}_1 := \{\text{maximal } R \in \mathcal{D}(Q) : \langle w\rangle_R > \alpha_1 = \frac{2^n}{\alpha}\langle w\rangle_Q\} = \{Q_j^1\}_j$; $U_1 := \bigcup_j Q_j^1$

$\vdots$

$\mathcal{S}_k := \{\text{maximal } R \in \mathcal{D}(Q) : \langle w\rangle_R > \alpha_k = \left(\frac{2^n}{\alpha}\right)^k\langle w\rangle_Q\} = \{Q_j^k\}_j$; $U_k := \bigcup_j Q_j^k$

Each collection satisfies:

1. $\boxed{\alpha_k < \langle w\rangle_{Q_j^k} \le 2^n \alpha_k}$ $\quad Q_j^k \in \mathcal{S}_k \Rightarrow$ parent $\widehat{Q_j^k}$ not chosen $\Rightarrow \langle w\rangle_{\widehat{Q_j^k}} \le \alpha_k \Rightarrow \langle w\rangle_{Q_j^k} \le \frac{2^n}{|\widehat{Q_j^k}|}\int_{\widehat{Q_j^k}} w \le 2^n\alpha_k$.

2. $\boxed{w(x) \le \alpha_k \text{ a.a. } x \notin U_k}$ $\quad x \notin U_k \Rightarrow \exists$ sequence $\{P_m\}$ of cubes s.t. $|P_m| \to 0$, $x \in P_m$, $P_m \notin \mathcal{S}_k \Rightarrow \langle w\rangle_{P_m} \le \alpha_k$
$\Rightarrow$ Lebesgue Differentiation Thm.: $w(x) = \lim_{m\to\infty}\langle w\rangle_{P_m} \le \alpha_k$ a.a. $x \notin U_k$.

3. $\boxed{\text{Each } Q_j^{k+1} \in \mathcal{S}_{k+1} \text{ is contained in some } Q_\ell^k \in \mathcal{S}_k.}$ $\quad Q_j^{k+1} \in \mathcal{S}_{k+1} \Rightarrow \langle w\rangle_{Q_j^{k+1}} > \alpha_{k+1} > \alpha_k \Rightarrow Q_j^{k+1}$ is contained in a maximal $Q_k^\ell \in \mathcal{S}_k$.

$\Rightarrow \boxed{U_0 \supset U_1 \supset U_2 \supset \dots}$

What portion of $Q_\ell^k \in \mathcal{S}_k$ is covered by U_{k+1}? $\quad \langle w\rangle_{Q_j^{k+1}} > \alpha_{k+1}$

$$2^n\alpha_k \ge \langle w\rangle_{Q_\ell^k} = \frac{1}{|Q_\ell^k|}\int_{Q_\ell^k} w \ge \frac{1}{|Q_\ell^k|}\int_{Q_\ell^k \cap U_{k+1}} w = \frac{1}{|Q_\ell^k|}\sum_{j: Q_j^{k+1} \subset Q_\ell^k}\int_{Q_j^{k+1}} w$$

$$> \frac{1}{|Q_\ell^k|}\sum_{j: Q_j^{k+1}\subset Q_\ell^k}\alpha_{k+1}|Q_j^{k+1}| = \alpha_{k+1}\frac{|Q_\ell^k \cap U_{k+1}|}{|Q_\ell^k|}$$

$\Rightarrow 2^n\alpha_k > \frac{2^n\alpha_k}{\alpha}\frac{|Q_\ell^k \cap U_{k+1}|}{|Q_\ell^k|} \Rightarrow \boxed{|Q_\ell^k \cap U_{k+1}| < \alpha|Q_\ell^k|} \to$ Summing over ℓ: $\boxed{|U_{k+1}| \le \alpha|U_k|}$

$\left.\begin{array}{l}\Rightarrow |U_k| \le \alpha^k|U_0| \\ \alpha \in (0,1)\end{array}\right\} \Rightarrow \lim_{k\to\infty}|U_k| = 0$

$\Rightarrow \left|\bigcap_{k=0}^{\infty} U_k\right| = 0$

$$\frac{w(Q_\ell^k \cap U_{k+1})}{w(Q_\ell^k)} < \beta = 1 - \frac{(1-\alpha)^p}{[w]_{A_p}}$$

$\Rightarrow$ Summing over ℓ: $\boxed{w(U_{k+1}) \le \beta\, w(U_k)}$

$\Rightarrow \boxed{w(U_k) \le \beta^k w(U_0)}$

Since $U_0 \supset U_1 \supset \dots$ and $\left|\bigcap_{k=0}^{\infty} U_k\right| = 0$, we can write

$$Q = (Q \setminus U_0) \cup (U_0 \setminus U_1) \cup \dots \quad \text{up to a null set}$$

$$\boxed{w(x) \le \alpha_k \text{ for a.a. } x \in Q \setminus U_k}$$

$$\int_Q w^{1+\gamma} = \int_{Q\setminus U_0} w^{\gamma} \cdot w + \int_{U_0 \setminus U_1} w^{\gamma} \cdot w + \dots + \int_{U_k \setminus U_{k+1}} w^{\gamma} \cdot w + \dots$$

$$\le \alpha_0^{\gamma} w(Q\setminus U_0) + \alpha_1^{\gamma} w(U_0 \setminus U_1) + \dots + \alpha_{k+1}^{\gamma} w(U_k \setminus U_{k+1}) + \dots$$

$$\boxed{w(U_k) \le \beta^k w(U_0)}$$

$$\le \alpha_0^{\gamma} w(Q\setminus U_0) + \alpha_1^{\gamma} w(U_0) + \alpha_2^{\gamma} w(U_1) + \dots + \alpha_{k+1}^{\gamma} w(U_k) + \dots$$

$$= \alpha_0^{\gamma} w(Q\setminus U_0) + \sum_{k=0}^{\infty} \alpha_{k+1}^{\gamma} w(U_k)$$

$$\le \alpha_0^{\gamma} w(Q\setminus U_0) + \sum_{k=0}^{\infty} \left(\frac{2^n}{\alpha}\right)^{(k+1)\gamma} \alpha_0^{\gamma} \, \beta^k w(U_0)$$

$$= \alpha_0^{\gamma} w(Q\setminus U_0) + \alpha_0^{\gamma} w(U_0) \left(\frac{2^n}{\alpha}\right)^{\gamma} \sum_{k=0}^{\infty} \left(\frac{2^n}{\alpha}\right)^{k\gamma} \beta^k$$

$$\le \underbrace{\alpha_0^{\gamma}}_{\langle w \rangle_Q^{\gamma}} \left(1 + \left(\frac{2^n}{\alpha}\right)^{\gamma} \sum_{k=0}^{\infty} \left(\frac{2^n}{\alpha}\right)^{k\gamma} \beta^k\right) w(Q) = \langle w \rangle_Q^{1+\gamma} \cdot |Q| \cdot \left(1 + \frac{\left(\frac{2^n}{\alpha}\right)^{\gamma}}{1 - \left(\frac{2^n}{\alpha}\right)^{\gamma} \beta}\right)$$

$$\sum_{k=0}^{\infty} \left[\left(\frac{2^n}{\alpha}\right)^{\gamma} \beta\right]^k < \infty \text{ as long as } \left(\frac{2^n}{\alpha}\right)^{\gamma} \beta < 1$$

and then this is $\dfrac{1}{1 - \left(\frac{2^n}{\alpha}\right)^{\gamma} \beta}$

$$\Rightarrow \left(\frac{1}{|Q|} \int_Q w^{1+\gamma}\right)^{\frac{1}{1+\gamma}} \le \langle w \rangle_Q \cdot \underbrace{\left(1 + \frac{\left(\frac{2^n}{\alpha}\right)^{\gamma}}{1 - \left(\frac{2^n}{\alpha}\right)^{\gamma} \beta}\right)^{\frac{1}{\gamma+1}}}_{C}$$

As long as γ is chosen small enough so $\left(\frac{2^n}{\alpha}\right)^{\gamma} \beta < 1$.

$$\left(\frac{2^n}{\alpha}\right)^{\gamma} < \frac{1}{\beta}$$

$$\gamma \log\left(\frac{2^n}{\alpha}\right) < -\log \beta$$

$$\gamma < \frac{-\log \beta}{\log(2^n) - \log \alpha} = \frac{\log\left(1 - \frac{(1-\alpha)^p}{[w]_{A_p}}\right)}{\log \alpha - \log(2^n)} = \frac{\log\left([w]_{A_p} - (1-\alpha)^p\right) - \log\left([w]_{A_p}\right)}{\log \alpha - \log(2^n)}$$

Choose $\alpha \in (0,1)$ to maximize the RHS. ■

Corollary 1: If $w \in A_p$, $1<p<\infty$, there exists $\gamma>0$ (depending on w, p, n) s.t. $\boxed{w^{1+\gamma} \in A_p}$

Proof: There are $\gamma_1, \gamma_2 > 0$ and $C_1, C_2 > 0$ s.t. Reverse Hölder holds for $w \in A_p$, $w' \in A_{p'}$:

$$\frac{1}{|Q|}\int_Q w^{1+\gamma_1} \leq \left(C_1 \langle w\rangle_Q\right)^{1+\gamma_1} \qquad \text{Take } \gamma := \min(\gamma_1, \gamma_2)$$

$$\frac{1}{|Q|}\int_Q (w')^{1+\gamma_2} \leq \left(C_2 \langle w'\rangle_Q\right)^{1+\gamma_2}$$

If $\boxed{\gamma_1 < \gamma_2}$: $p_1 = \frac{1+\gamma_2}{1+\gamma_1} > 1$ => can do Hölder w/ p_1 and $p_1' = \frac{p_1}{p_1 - 1}$

$$\frac{1}{|Q|}\int_Q (w')^{1+\gamma_1} \leq \frac{1}{|Q|}\left(\int_Q (w')^{1+\gamma_2}\right)^{\frac{1+\gamma_1}{1+\gamma_2}} |Q|^{1-\frac{1}{p_1}} = \left(\frac{1}{|Q|}\int_Q (w')^{1+\gamma_2}\right)^{\frac{1+\gamma_1}{1+\gamma_2}} \leq \left(\left(C_2\langle w'\rangle_Q\right)^{1+\gamma_2}\right)^{\frac{1+\gamma_1}{1+\gamma_2}}$$

$$\boxed{\Rightarrow \text{if } \gamma_1 < \gamma_2: \quad \frac{1}{|Q|}\int_Q (w')^{1+\gamma_1} \leq \left(C_2 \langle w'\rangle_Q\right)^{1+\gamma_1}}$$

If $\boxed{\gamma_2 < \gamma_1}$: $p_1 = \frac{1+\gamma_1}{1+\gamma_2} > 1$ => Hölder with p_1 & $p_1' = \frac{p_1}{p_1 - 1}$

$$\frac{1}{|Q|}\int_Q w^{1+\gamma_2} \leq \frac{1}{|Q|}\left(\int_Q w^{1+\gamma_1}\right)^{\frac{1+\gamma_2}{1+\gamma_1}} |Q|^{1-1/p_1} = \left(\frac{1}{|Q|}\int_Q w^{1+\gamma_1}\right)^{\frac{1+\gamma_2}{1+\gamma_1}} \leq \left(\left(C_1\langle w\rangle_Q\right)^{1+\gamma_1}\right)^{\frac{1+\gamma_2}{1+\gamma_1}}$$

$$\Rightarrow \text{if } \gamma_2 < \gamma_1: \quad \frac{1}{|Q|}\int_Q w^{1+\gamma_2} \leq \left(C_1 \langle w\rangle_Q\right)^{1+\gamma_2}$$

=> If $\gamma := \min(\gamma_1, \gamma_2)$, both hold with γ:

$$\frac{1}{|Q|}\int_Q w^{1+\gamma} \leq \left(C_1 \langle w\rangle_Q\right)^{1+\gamma}$$

$$\frac{1}{|Q|}\int_Q (w')^{1+\gamma} \leq \left(C_2 \langle w'\rangle_Q\right)^{1+\gamma}$$

$$\Rightarrow \langle w^{1+\gamma}\rangle_Q \left\langle \left(w^{1+\gamma}\right)^{-\frac{1}{p-1}}\right\rangle_Q^{p-1} \leq \left(C_1 \langle w\rangle_Q C_2^{p-1} \langle w'\rangle_Q^{p-1}\right)^{1+\gamma} \leq \left(C_1 C_2^{p-1}\right)^{1+\gamma} [w]_{A_p}^{1+\gamma}$$

$$\Rightarrow w^{1+\gamma} \in A_p, \quad [w^{1+\gamma}]_{A_p} \leq \left(C_1 C_2^{p-1}\right)^{1+\gamma} [w]_{A_p}^{1+\gamma} \qquad \blacksquare$$

Small Lemma: Let $w \in A_p$ and $\delta \in (0,1)$. Then $\boxed{w^{\delta} \in A_q}$, where $\boxed{q = \delta p + 1 - \delta}$, with:

$$\boxed{[w^{\delta}]_{A_q} \leq [w]_{A_p}^{\delta}}$$

$$\langle w^{\delta}\rangle_Q \left\langle (w^{\delta})^{-\frac{1}{q-1}}\right\rangle_Q^{q-1} = \langle w^{\delta}\rangle_Q \left\langle w^{\frac{-\delta}{\delta p - \delta}}\right\rangle_Q^{\delta(p-1)} = \langle w^{\delta}\rangle \langle w'\rangle_Q^{\delta(p-1)} \leq \langle w\rangle_Q^{\delta} \langle w'\rangle_Q^{\delta(p-1)} \leq [w]_{A_p}^{\delta}.$$

$$\langle w^{\delta}\rangle_Q \leq \frac{1}{|Q|}\left(\int_Q w\right)^{\delta} |Q|^{1-\delta} = \langle w\rangle_Q^{\delta}$$

$\delta \in (0,1) \Rightarrow p_1 = \frac{1}{\delta} > 1$; $p_1' = \frac{\frac{1}{\delta}}{\frac{1}{\delta}-1} = \frac{1}{1-\delta}$

Corollary 2: For every $w \in A_p$, $1<p<\infty$, there is $q<p$ such that $\boxed{w \in A_q}$

Proof: Take the setup in Corollary 1: γ, c_1, c_2 s.t.

$$w^{1+\gamma} \in A_p, \quad [w^{1+\gamma}]_{A_p} \leq (c_1 c_2^{p-1})^{1+\gamma} [w]_{A_p}^{1+\gamma}$$

Let $\delta = \frac{1}{1+\gamma} \in (0,1) \Rightarrow (w^{1+\gamma})^{\delta} = w \in A_q$, where $q = \frac{1}{1+\gamma} p + 1 - \frac{1}{1+\gamma} = \frac{p+\gamma}{1+\gamma}$ with

$$[w]_{A_q} \leq [w^{1+\delta}]_{A_p}^{\delta} \leq \left((c_1 c_2^{p-1})^{1+\gamma} [w]_{A_p}^{1+\gamma}\right)^{\frac{1}{1+\gamma}} = (c_1 c_2^{p-1}) [w]_{A_p}$$

The claim follows since:

$$q = \frac{p+\gamma}{1+\gamma} < p$$

$$\cancel{p} + \gamma < \cancel{p} + p\gamma$$

$$1 < p$$

$\blacksquare$